

LOCAL TROPICAL VARIETY

NAOYUKI TOUDA

1. INTRODUCTION

The tropical variety of an ideal of $K[x] := K[x_1, \dots, x_n]$, where $K = \mathbb{C}\{\{t\}\}$ is the field of Puiseux series, was introduced by D. Speyer and B. Sturmfels [12]. Let us sketch their construction. Any Puiseux series $p(t)$ can be written as $p(t) = c_1 t^{q_1} + c_2 t^{q_2} + c_3 t^{q_3} + \dots$, where c_1, c_2, \dots are non-zero complex numbers and $q_1 < q_2 < \dots$ are rational numbers with common denominator. The order of $p(t)$, denoted by $\text{ord}(p(t))$, is the exponent q_1 . Then we have the following map:

$$\begin{aligned} \text{ord} : (K \setminus \{0\})^n &\rightarrow \mathbb{Q}^n \subset \mathbb{R}^n \\ \text{ord}(p_1(t), \dots, p_n(t)) &\mapsto (\text{ord}(p_1(t)), \dots, \text{ord}(p_n(t))) \end{aligned}$$

We fix a weight vector $w \in \mathbb{R}^n$. The weight of the variable x_i is w_i . The weight of a term $p_i(t)x_1^{a_{i1}} \dots x_n^{a_{in}}$ is the real number $\text{ord}(p_i(t)) + a_{i1}w_1 + \dots + a_{in}w_n$. Consider a polynomial $f = \sum p_i(t)x_1^{a_{i1}} \dots x_n^{a_{in}} \in K[x]$. Let \bar{w} be the smallest weight among all the terms in f . The initial form of f equals

$$\text{in}_w(f) = \sum c_{\alpha_1, \dots, \alpha_n} \cdot x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

where the sum ranges over all the terms $p_i(t)x_1^{a_{i1}} \dots x_n^{a_{in}}$ in f whose w -weight coincides with \bar{w} and where $c_{\alpha_1, \dots, \alpha_n} \in \mathbb{C}$ denotes the coefficient of $t^{\bar{w} - \alpha_1 w_1 - \dots - \alpha_n w_n}$ in the Puiseux series $p(t)$. The initial ideal $\text{in}_w(I) \subset K[x]$ is defined as the ideal generated by all the initial forms $\text{in}_w(f)$ where f runs over I .

Given an ideal I in $K[x_1, \dots, x_n]$. We define its zero set $V(I) := \{v \in (K \setminus \{0\})^n \mid f(v) = 0, \forall f \in I\}$. Then the tropical variety $\mathcal{T}(I)$ is defined as $\mathcal{T}(I) = \overline{\text{ord}(V(I))} \subset \mathbb{R}^n$, the topological closure of the image of $V(I)$ under the map above.

In tropical geometry we consider the tropical semiring $(\mathbb{R} \cup \{+\infty\}, \oplus, \odot)$, which has tropical addition \oplus and tropical multiplication \odot , that means, for $x, y \in \mathbb{R} \cup \{+\infty\}$, $x \oplus y := \min\{x, y\}$ and $x \odot y := x + y$. Then for vectors $(a_1, \dots, a_n), (b_1, \dots, b_n) \in \mathbb{R}^n$, its tropical addition is $(a_1, \dots, a_n) \oplus (b_1, \dots, b_n) := (\min\{a_1, b_1\}, \dots, \min\{a_n, b_n\})$ and its tropical scalar multiplication is $\lambda \odot (a_1, a_2, \dots, a_n) := (\lambda + a_1, \lambda + a_2, \dots, \lambda + a_n)$ ($\lambda \in \mathbb{R}$). A tropical monomial is an expression of the form $c \odot x_1^{a_1} \odot \dots \odot x_n^{a_n}$, $c \in \mathbb{R}$ where the powers of the variables are computed tropically as well, for instance, $x_1^3 = x_1 \odot x_1 \odot x_1$. A tropical polynomial is a finite tropical sum of tropical monomials, $F = c_1 \odot x_1^{a_{11}} \odot \dots \odot x_n^{a_{1n}} \oplus \dots \oplus c_r \odot x_1^{a_{r1}} \odot \dots \odot x_n^{a_{rn}}$. Then for any $(w_1, \dots, w_n) \in \mathbb{R}^n$, $F(w_1, \dots, w_n) = \min_{1 \leq i \leq r} \{c_i + a_{i1}w_1 + \dots + a_{in}w_n\}$. The tropical hypersurface of F , denoted by $\mathcal{T}_h(F)$, is the set of points $(w_1, \dots, w_n) \in \mathbb{R}^n$ at which the minimum of $F(w_1, \dots, w_n)$ is attained twice or more. For a polynomial $f = \sum_{i=1}^r p_i(t)x_1^{a_{i1}} \dots x_n^{a_{in}}$ ($p_i(t) \in K$) in $K[x]$,

$\text{trop}(f) := c_1 \odot x_1^{a_{11}} \odot \cdots \odot x_n^{a_{1n}} \oplus \cdots \oplus c_r \odot x_1^{a_{r1}} \odot \cdots \odot x_n^{a_{rn}}$, ($c_i = \text{ord}(p_i(t))$) is called the tropicalization of f .

With the notations above, the following theorem was proved by D.Speyer and B.Sturmfels:

Theorem 1.1. [[12] Theorem 2.1.] For any ideal I in $K[x]$, the following subsets of \mathbb{R}^n coincide:

- (1) $\mathcal{T}(I) = \overline{\text{ord}(V(I))}$;
- (2) $\bigcap_{f \in I} \mathcal{T}_h(\text{trop}(f))$;
- (3) $\{w \in \mathbb{R}^n \mid \text{for any } f \in I, \text{ in}_w(f) \text{ is not a monomial}\}$.

In the particular case $K = \mathbb{C}$, we have:

Proposition 1.2. Let I be a homogeneous ideal in $\mathbb{C}[x]$, then $\mathcal{T}(I)$ is a subfan of the Gröbner fan $\text{GF}(I)$.

In this paper, we will give an analogous definition of local tropical varieties and prove analogous theorems in the formal power series ring $\mathbb{C}[[x_1, \dots, x_n]] = \hat{\mathcal{O}}$.

2. LOCAL TROPICAL HYPERSURFACE

In this section we will define local tropical hypersurface. First, let's introduce some notations which are necessary for our local construction.

We denote by \mathcal{U}_{loc} the set of local weights $\{u \in \mathbb{R}^n \mid u_i \geq 0, \forall i\}$. For $f = \sum_{\alpha} p_{\alpha} x^{\alpha} \in \hat{\mathcal{O}}$, we define its support $\text{Supp}(f) \subset \mathbb{N}^n$ as the set of α such that $p_{\alpha} \neq 0$. For a local weight vector $u \in \mathcal{U}_{\text{loc}}$, the u -weight of f (denoted by $\text{wei}^u(f)$) is the minimum of the scalar products $u \cdot \alpha$ for $\alpha \in \text{Supp}(f)$. The weight gives rise to a filtration $F^u(\hat{\mathcal{O}})$ given by $F_k^u := \{f \mid \text{wei}^u(f) \geq k\}$ as well as to the associated graded ring $\text{Gr}^u(\hat{\mathcal{O}}) := \bigoplus_k F_k^u / F_{k>}^u$ (direct sum). For $f \in \hat{\mathcal{O}}$, its initial form $\text{in}_u(f)$ shall be the class of f in $F_k^u / F_{k>}^u$ where $k = \text{wei}^u(f)$. If $I \subset \hat{\mathcal{O}}$ is an ideal, then its initial ideal $\text{in}_u(I)$ is an ideal in $\text{Gr}^u(\hat{\mathcal{O}})$ generated by all the $\text{in}_u(f)$ for $f \in I$, in other words, $\text{in}_u(I) := \langle \text{in}_u(f) \mid f \in I \rangle_{\text{Gr}^u(\hat{\mathcal{O}})}$.

Lemma 2.1. Let $u \in \mathcal{U}_{\text{loc}}$ be a local weight vector and suppose that $u = (u_1, \dots, u_m, 0, \dots, 0)$ with $0 \leq m \leq n$ and $u_i > 0$. Then

$$\text{Gr}^u(\hat{\mathcal{O}}) = \mathbb{C}[x_1, \dots, x_m][[x_{m+1}, \dots, x_n]].$$

Thanks to the preceding lemma, we shall see $\text{Gr}^u(\hat{\mathcal{O}})$ as a subring of $\hat{\mathcal{O}}$.

Definition 2.2. For $f = \sum_{\alpha} p_{\alpha} x^{\alpha} \in \hat{\mathcal{O}}$, we define the tropicalization of f as

$$\text{trop}(f) := \bigoplus_{\alpha \in \text{Supp}(f)} a_{\alpha} \odot x^{\alpha} \quad (\text{infinite tropical sum})$$

and its tropical hypersurface as

$$\mathcal{T}_{\text{loc}}^h(\text{trop}(f)) := \{w \in \mathcal{U}_{\text{loc}} \mid \text{the minimum is attained twice or more}\}.$$

The following proposition, which is analogous to the case of the polynomial ring, holds:

Proposition 2.3. *For any ideal $I \subset \hat{\mathcal{O}}$, the following subsets of \mathcal{U}_{loc} coincide:*

- (1) $\{w \in \mathcal{U}_{\text{loc}} \mid \forall f \in I, \text{in}_w(f) \in \text{Gr}^w(\hat{\mathcal{O}}) \text{ is not a monomial}\};$
- (2) $\bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(f)).$

Proof. The proof is analogous to the case of polynomial ring, [12, Theorem 2.1]. \square

3. LOCAL TROPICAL VARIETY (PRINCIPAL CASE)

In this section we will define the local tropical variety of a principal ideal in $\hat{\mathcal{O}}$. For this purpose, the following proposition will play an important role.

Proposition 3.1. *For $f = \sum_{\alpha \in \text{Supp}(f)} p_\alpha x^\alpha \in \mathbb{C}[[x]]$, there exists the finite and minimal subset $E_0(f) \subset \text{Supp}(f)$ s.t.*

$$E_0(f) + \mathbb{N}^n = \text{Supp}(f) + \mathbb{N}^n.$$

Now we will construct the local tropical variety of a principal ideal in $\hat{\mathcal{O}}$.

3.1. Tropical Variety on the Maximal Stratum. In this subsection, we suppose w lies in $\mathcal{U}_{\text{loc}}^0 := \{(u_1, \dots, u_n) \mid \forall i, u_i > 0\}$.

Definition 3.2. *Let $f = \sum_{\alpha \in \text{Supp}(f)} p_\alpha x^\alpha \in \mathbb{C}[[x]]$, then we define*

$$\tilde{f}^0 := \sum_{\beta \in E_0(f)} p_\beta x^\beta \in \mathbb{C}[x] \subset \mathbb{C}[[x]].$$

Proposition 3.3. *The following subsets of $\mathcal{U}_{\text{loc}}^0$ coincide:*

- (1) $\{w \in \mathcal{U}_{\text{loc}}^0 \mid \forall f \in I, \text{in}_w(f) \in \text{Gr}^w(K[[x]]) \text{ is not a monomial}\};$
- (2) $\mathcal{T}_{\text{loc}}^h(\text{trop}(f)) \cap \mathcal{U}_{\text{loc}}^0;$
- (3) $\mathcal{T}_{\text{loc}}^h(\text{trop}(\tilde{f}^0)) \cap \mathcal{U}_{\text{loc}}^0;$
- (4) $\overline{\text{ord}(V(\langle \tilde{f}^0 \rangle))} \cap \mathcal{U}_{\text{loc}}^0.$

Here, $V(\langle \tilde{f}^0 \rangle)$ is defined by $\{v \in (\mathbb{C}\{\{t\}\} \setminus \{0\})^n \mid \tilde{f}^0(v) = 0\}$.

To prove the previous proposition, we will use the next lemma:

Lemma 3.4. *Fix $u = (u_1, \dots, u_n) \in \mathcal{U}_{\text{loc}}^0$. Suppose that $(\alpha_1, \dots, \alpha_n) \in \text{Supp}(f)$ satisfies $\alpha_1 u_1 + \dots + \alpha_n u_n \leq \alpha'_1 u_1 + \dots + \alpha'_n u_n$ for any $(\alpha'_1, \dots, \alpha'_n) \in \text{Supp}(f)$. Then $(\alpha_1, \dots, \alpha_n)$ is in $E_0(f)$.*

The proof is easy.

Proof of Proposition 3.3. Proofs of (1)=(2) and (3)=(4) is similar to the case of polynomial ring. See [12, Theorem 2.1].

((2) \subset (3)) : Take $w = (w_1, \dots, w_n) \in (2)$. Then there exist $\alpha_1, \alpha_2 \in \text{Supp}(f)$ satisfying $\alpha_1 \cdot w = \alpha_2 \cdot w \leq \alpha \cdot w$ for any $\alpha \in \text{Supp}(f)$. By Lemma 3.4, this implies $\alpha_1, \alpha_2 \in E_0(f)$. Since $E_0(f) \subset \text{Supp}(f)$, $w \in (3)$.

((3) \subset (2)) : Take $w = (w_1, \dots, w_n) \in (3)$. Then there exist $\beta_1, \beta_2 \in E_0(f)$ satisfying $\beta_1 \cdot w = \beta_2 \cdot w \leq \beta \cdot w$ for any $\beta \in E_0(f)$. By Lemma 3.4, this implies $\beta_1 \cdot w = \beta_2 \cdot w \leq \alpha \cdot w$ for any $\alpha \in \text{Supp}(f)$. \square

3.2. Tropical Variety on a substratum $\mathcal{U}_{\text{loc}}^1$. In this subsection, we suppose that w lies in $\mathcal{U}_{\text{loc}}^1 := \{(0, u_2, \dots, u_n) \mid \forall i (\neq 1), u_i > 0\}$.

First, we consider the following map:

$$\begin{array}{ccc} \phi_1 : \hat{\mathcal{O}} = \mathbb{C}[[x_1, \dots, x_n]] & \rightarrow & (\mathbb{C}[[x_1]])[[x_2, \dots, x_n]]. \\ f & \mapsto & \phi_1(f) \end{array}$$

That means, we consider x_1 as a parameter and $f = \phi_1(f)$ as an element of $(\mathbb{C}[[x_1]])[[x_2, \dots, x_n]]$, the formal power series ring with the variables x_2, \dots, x_n over the formal power series ring $\mathbb{C}[[x_1]]$. Then we define $E_1(\phi_1(f)) \subset \mathbb{N}^{n-1}$ as in Proposition 3.1. We denote by Φ_1 the projection map from the set of points $\text{Supp}(f)$ to $\mathbb{N}^{n-1} \cap \text{Supp}(f)$. Define $E_1(f) (\subset \text{Supp}(f))$ with $\Phi_1^{-1}(E_1(\phi_1(f)))$.

Definition 3.5. $\tilde{f}^1 := \sum_{\beta \in E_1(f)} a_\beta x^\beta \in \hat{\mathcal{O}}$.

Then, as an analogy of Lemma 3.4, we have:

Lemma 3.6. Fix $u = (0, u_2, \dots, u_n) \in \mathcal{U}_{\text{loc}}^1$. Suppose that $(\alpha_1, \dots, \alpha_n) \in \text{Supp}(f)$ satisfies $\alpha_1 \cdot 0 + \alpha_2 \cdot u_2 + \dots + \alpha_n \cdot u_n \leq \alpha'_1 \cdot 0 + \alpha'_2 \cdot u_2 + \dots + \alpha'_n \cdot u_n$ for any $(\alpha'_1, \dots, \alpha'_n) \in \text{Supp}(f)$. Then $(\alpha_1, \dots, \alpha_n)$ is in $E_1(f)$.

Now, we suppose $E_1(\phi(f)) = \{\beta^1 = (\beta_2^1, \dots, \beta_n^1), \dots, \beta^l = (\beta_2^l, \dots, \beta_n^l)\}$. Then $E_1(f) = \{E_1^1, \dots, E_1^l\}$, where $\Phi_1^{-1}(\beta^i) = E_1^i$ ($1 \leq i \leq l$).

Lemma 3.7. Take $\alpha^1, \alpha^2 \in E_1^i$, ($i = 1, \dots, l$). For any $w \in \mathcal{U}_{\text{loc}}^1$, $\alpha^1 \cdot w = \alpha^2 \cdot w$.

Now, we set $E_1(f) = \{E_1^1, \dots, E_1^l\}$ as above. For each E^i , if $\#(E_1^i) \geq 2$, we choose two elements α^1, α^2 ($\alpha^1 \neq \alpha^2$) from E_1^i and we set $A_1^i = \{\alpha^1, \alpha^2\}$. If $\#(E_1^i) = 1$, we set $A_1^i = E_1^i$. Then, we define $A_1(f) = \{A_1^1, \dots, A_1^l\} (\subset E_1(f))$. Clearly, $A_1(f)$ is a finite set ($\#(A_i(f)) \leq 2l$).

Definition 3.8. $\hat{f}^1 := \sum_{\alpha \in A_1(f)} a_\alpha x^\alpha \in \mathbb{C}[x]$

Proposition 3.9. $\mathcal{T}_{\text{loc}}^h(\text{trop}(\tilde{f}^1)) \cap \mathcal{U}_{\text{loc}}^1 = \mathcal{T}_{\text{loc}}^h(\text{trop}(\hat{f}^1)) \cap \mathcal{U}_{\text{loc}}^1$

Proof. (\subset): Take $w \in \text{LHS}$. Then there exist $b \in E_1^i$ and $c \in E_1^j$ ($1 \leq i, j \leq l$) s.t. $b \cdot w = c \cdot w \leq a \cdot w$ for any $a \in E_1(f)$. By Lemma 3.7, there exist $\hat{b} \in A_1^i$ and $\hat{c} \in A_1^j$ s.t. $b \cdot w = \hat{b} \cdot w$, $c \cdot w = \hat{c} \cdot w$. Since $A_1(f) \subset E_1(f)$, this implies $w \in \text{RHS}$.

(\supset): Take $w \in \text{RHS}$. Then there exist $b \in A_1^i$ and $c \in A_1^j$ ($1 \leq i, j \leq l$) s.t. $b \cdot w = c \cdot w \leq a \cdot w$ for any $a \in A_1(f)$. Now we suppose that there exists $d \in E_1^k (\subset E_1(f))$ s.t. $d \cdot w \not\leq b \cdot w = c \cdot w$. Then, by Lemma 3.7, for any $\hat{d} \in A_1^k$, $\hat{d} \cdot w = d \cdot w \not\leq b \cdot w = c \cdot w$. This contradicts the hypothesis. Thus, $b \cdot w = c \cdot w \leq a \cdot w$ for any $a \in E_1(f)$ and this implies $w \in \text{LHS}$. \square

Analogously to the Proposition 3.3, the following proposition holds:

Proposition 3.10. The following subsets of $\mathcal{U}_{\text{loc}}^1$ coincide:

- (1) $\{w \in \mathcal{U}_{\text{loc}}^1 \mid \forall f \in I, \text{in}_w(f) \in \text{Gr}^w(\hat{\mathcal{O}}) \text{ is not a monomial}\};$
- (2) $\mathcal{T}_{\text{loc}}^h(\text{trop}(f)) \cap \mathcal{U}_{\text{loc}}^1;$

- (3) $\mathcal{T}_{\text{loc}}^h(\text{trop}(\tilde{f}^1)) \cap \mathcal{U}_{\text{loc}}^1$;
- (4) $\mathcal{T}_{\text{loc}}^h(\text{trop}(\hat{f}^1)) \cap \mathcal{U}_{\text{loc}}^1$;
- (5) $\text{ord}(V(\langle \hat{f}^1 \rangle)) \cap \mathcal{U}_{\text{loc}}^1$.

Proof. The proofs of (1)=(2) and (4)=(5) are analogous to the proof of the polynomial case. By Proposition 3.9, it is enough to prove the equality of (2) and (3).

((2) \subset (3)): Take $w = (0, w_2, \dots, w_n) \in (2)$. Then there exist $\alpha_1, \alpha_2 \in \text{Supp}(f)$ s.t. $\alpha_1 \cdot w = \alpha_2 \cdot w \leq \alpha \cdot w$ for any $\alpha \in \text{Supp}(f)$. By Lemma 3.7, this implies $\alpha_1, \alpha_2 \in E_1(f)$. Since $E_1(f) \subset \text{Supp}(f)$, we have $w \in (3)$.

((3) \subset (2)) : Take $w = (0, w_2, \dots, w_n) \in (3)$ Then there exist $\beta_1, \beta_2 \in E_1(f)$ s.t. $\beta_1 \cdot w = \beta_2 \cdot w \leq \beta \cdot w$ for any $\beta \in E_1(f)$. By Lemma 3.7, $\beta_1 \cdot w = \beta_2 \cdot w \leq \alpha \cdot w$ for any $\alpha \in \text{Supp}(f)$. This implies $w \in (2)$. \square

3.3. Tropical Variety on General Strata. In this subsection, we suppose that w lies in \mathcal{U}_{loc} .

We will use the same argument as in 3.2. For $\mathcal{U}_{\text{loc}}^j := \{(u_1, \dots, u_n) \mid \forall i (\neq j), u_i > 0, u_j = 0\}, \dots, \mathcal{U}_{\text{loc}}^n, \mathcal{U}_{\text{loc}}^{12} := \{(0, 0, u_3, \dots, u_n) \mid \forall i (\neq 1, 2), u_i > 0\}, \dots, \mathcal{U}_{\text{loc}}^{234 \dots n} := \{(0, \dots, 0, u_n) \mid u_n > 0, \forall j (\neq n), u_j = 0\}, \mathcal{U}_{\text{loc}}^{123 \dots n} := \{(0, \dots, 0)\}$.

Definition 3.11. For $\mathcal{U}_{\text{loc}}^{123 \dots n} := \{(0, \dots, 0)\}$, we define $\tilde{f}^{12 \dots n} = \hat{f}^{12 \dots n} := 0$. For $\mathcal{U}_{\text{loc}}^0 := \{(u_1, \dots, u_n) \mid \forall i, u_i > 0\}$, we define $\tilde{f}^0 = \hat{f}^0$.

Then, for each $\lambda \in \Lambda = \{0, 1, 2, \dots, n, 12, 13, \dots, 234 \dots n, 123 \dots n\}$, the following proposition holds:

Proposition 3.12. The following subsets of $\mathcal{U}_{\text{loc}}^\lambda$ coincide:

- (1) $\{w \in \mathcal{U}_{\text{loc}}^\lambda \mid \forall f \in I, \text{in}_w(f) \in \text{Gr}^w(\hat{\mathcal{O}}) \text{ is not a monomial}\}$;
- (2) $\mathcal{T}_{\text{loc}}^h(\text{trop}(f)) \cap \mathcal{U}_{\text{loc}}^\lambda$;
- (3) $\mathcal{T}_{\text{loc}}^h(\text{trop}(\tilde{f}^\lambda)) \cap \mathcal{U}_{\text{loc}}^\lambda$;
- (4) $\mathcal{T}_{\text{loc}}^h(\text{trop}(\hat{f}^\lambda)) \cap \mathcal{U}_{\text{loc}}^\lambda$;
- (5) $\text{ord}(V(\langle \hat{f}^\lambda \rangle)) \cap \mathcal{U}_{\text{loc}}^\lambda$.

Now, we have the following theorem:

Theorem 3.13. The following subsets of \mathcal{U}_{loc} coincide:

- (1) $\{w \in \mathcal{U}_{\text{loc}} \mid \forall f \in I, \text{in}_w(f) \in \text{Gr}^w(\hat{\mathcal{O}}) \text{ is not a monomial}\}$;
- (2) $\mathcal{T}_{\text{loc}}^h(\text{trop}(f))$;
- (3) $\bigcup_{\lambda \in \Lambda} (\mathcal{T}_{\text{loc}}^h(\text{trop}(\tilde{f}^\lambda)) \cap \mathcal{U}_{\text{loc}}^\lambda)$;
- (4) $\bigcup_{\lambda \in \Lambda} (\mathcal{T}_{\text{loc}}^h(\text{trop}(\hat{f}^\lambda)) \cap \mathcal{U}_{\text{loc}}^\lambda)$;
- (5) $\bigcup_{\lambda \in \Lambda} (\text{ord}(V(\langle \hat{f}^\lambda \rangle)) \cap \mathcal{U}_{\text{loc}}^\lambda)$.

In view of this theorem, we will define our local tropical variety of f as the union of the tropical varieties defined on strata $\mathcal{U}_{\text{loc}}^\lambda$ for finite polynomials \hat{f} .

Definition 3.14. Let $I = \langle f \rangle_{\hat{\mathcal{O}}}$ be a principal ideal for some $f \in \hat{\mathcal{O}}$. Then we define the local tropical variety of I as

$$\mathcal{T}_{\text{loc}}(I) := \bigcup_{\lambda \in \Lambda} \left(\overline{\text{ord}(V(\langle \hat{f}^\lambda \rangle))} \cap \mathcal{U}_{\text{loc}}^\lambda \right).$$

The following proposition shows that our definition is compatible with that of polynomial tropical varieties.

Proposition 3.15. For $f \in \mathbb{C}[x] \subset \hat{\mathcal{O}}$, let $I = \langle f \rangle_{\mathbb{C}[x]}$ and $I^e = \langle f \rangle_{\hat{\mathcal{O}}}$. Then $\mathcal{T}(I) \cap \mathcal{U}_{\text{loc}}$ and $\mathcal{T}_{\text{loc}}(I^e)$ are equal as sets.

Proof. It follows from Theorem 1.1 that we have $\mathcal{T}(I) \cap \mathcal{U}_{\text{loc}} = \mathcal{T}_h(\text{trop}(f)) \cap \mathcal{U}_{\text{loc}} = \mathcal{T}_{\text{loc}}^h(\text{trop}(f))$. By Theorem 3.13,

$$\mathcal{T}_{\text{loc}}^h(\text{trop}(f)) = \bigcup_{\lambda \in \Lambda} \left(\overline{\text{ord}(V(\langle \hat{f}^\lambda \rangle))} \cap \mathcal{U}_{\text{loc}}^\lambda \right) = \mathcal{T}_{\text{loc}}(I^e).$$

□

We have introduced auxiliary polynomials \hat{f} to define a local tropical variety with utilizing results on polynomial tropical varieties. It is an open question to give a description of our local tropical zero set without introducing the auxiliary polynomials \hat{f} . For example, we conjecture that we do not need \hat{f} to define our local tropical zero set on the maximal stratum. In fact, let us introduce the set of Puiseux series of positive order

$$K_+ = \{p(t) \in K = \mathbb{C}\{\{t\}\} \mid \text{ord}(p) > 0\}.$$

Note that the composition of f and $p \in K_+^n$ is well-defined. We conjecture that

$$\overline{\text{ord}(V(\langle \tilde{f}^0 \rangle))} \cap \mathcal{U}_{\text{loc}}^0 = \overline{\text{ord}(V_+(f))} \cap \mathcal{U}_{\text{loc}}^0$$

where

$$V_+(f) = \{p(t) = (p_1(t), \dots, p_n(t)) \in K_+^n \mid f(p) = 0\}.$$

4. LOCAL TROPICAL VARIETY (GENERAL CASE)

Suppose that $I \subset \hat{\mathcal{O}}$ is any ideal. In this section, most of the proofs follow from the principal ideal case.

4.1. Tropical Variety on the Maximal Stratum. In this subsection, we suppose that w lies in $\mathcal{U}_{\text{loc}}^0 := \{(u_1, \dots, u_n) \mid \forall i, u_i > 0\}$.

For each $f \in I$, similarly to Proposition 3.1, we define $E_0(f) \subset \text{Supp}(f)$. Then, we set $\tilde{f}^0 := \sum_{\beta \in E_0(f)} a_\beta x^\beta \in \mathbb{C}[x]$.

Proposition 4.1. The following subsets of $\mathcal{U}_{\text{loc}}^0$ coincide:

- (1) $\{w \in \mathcal{U}_{\text{loc}}^0 \mid \text{For any } f \in I, \text{in}_w(f) \in \text{Gr}^w(\hat{\mathcal{O}}) \text{ is not a monomial}\};$
- (2) $(\bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(f))) \cap \mathcal{U}_{\text{loc}}^0;$
- (3) $(\bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(\tilde{f}^0))) \cap \mathcal{U}_{\text{loc}}^0;$
- (4) $(\bigcap_{f \in I} \overline{\text{ord}(V(\langle \tilde{f}^0 \rangle))}) \cap \mathcal{U}_{\text{loc}}^0.$

4.2. Tropical Variety on a substratum $\mathcal{U}_{\text{loc}}^1$. In this subsection, we suppose that w lies in $\mathcal{U}_{\text{loc}}^1 := \{(0, u_2, \dots, u_n) \mid \forall i (\neq 1), u_i > 0\}$.

Similarly to Definitions 3.5 and 3.8, for each $f \in I$, we define:

Definition 4.2. $\tilde{f}^1 := \sum_{\beta \in E_1(f)} a_\beta x^\beta$ and $\widehat{f}^1 := \sum_{\beta \in A_1(f)} a_\beta x^\beta$.

Proposition 4.3. *The following subsets of $\mathcal{U}_{\text{loc}}^1$ coincide:*

- (1) $\{w \in \mathcal{U}_{\text{loc}}^1 \mid \text{For any } f \in I, \text{in}_w(f) \in \text{Gr}^w(\hat{\mathcal{O}}) \text{ is not a monomial}\};$
- (2) $\left(\bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(f))\right) \cap \mathcal{U}_{\text{loc}}^1;$
- (3) $\left(\bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(\tilde{f}^1))\right) \cap \mathcal{U}_{\text{loc}}^1;$
- (4) $\left(\bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(\widehat{f}^1))\right) \cap \mathcal{U}_{\text{loc}}^1;$
- (5) $\left(\bigcap_{f \in I} \overline{\text{ord}(V(\widehat{f}^1))}\right) \cap \mathcal{U}_{\text{loc}}^1.$

4.3. Tropical Variety on General Strata. In this subsection, we suppose that w lies in \mathcal{U}_{loc} .

We will use the same argument as in 4.2. For $\mathcal{U}_{\text{loc}}^j := \{(u_1, \dots, u_n) \mid \forall i (\neq j), u_i > 0, u_j = 0\}, \dots, \mathcal{U}_{\text{loc}}^n, \mathcal{U}_{\text{loc}}^{12} := \{(0, 0, u_3, \dots, u_n) \mid \forall i (\neq 1, 2), u_i > 0\}, \dots, \mathcal{U}_{\text{loc}}^{234 \dots n} := \{(0, \dots, 0, u_n) \mid u_n > 0, \forall j (\neq n), u_j = 0\}, \mathcal{U}_{\text{loc}}^{123 \dots n} := \{(0, \dots, 0)\}.$

Definition 4.4. For $\mathcal{U}_{\text{loc}}^{123 \dots n} := \{(0, \dots, 0)\}$, we define $\tilde{f}^{12 \dots n} = \widehat{f}^{12 \dots n} := 0$. For $\mathcal{U}_{\text{loc}}^0 := \{(u_1, \dots, u_n) \mid \forall i, u_i > 0\}$, we define $\tilde{f}^0 = \widehat{f}^0$.

Then, for each $\lambda \in \Lambda = \{0, 1, 2, \dots, n, 12, 13, \dots, 234 \dots n, 123 \dots n\}$, the following proposition holds:

Proposition 4.5. *The following subsets of $\mathcal{U}_{\text{loc}}^\lambda$ coincide:*

- (1) $\{w \in \mathcal{U}_{\text{loc}}^\lambda \mid \text{For any } f \in I, \text{in}_w(f) \in \text{Gr}^w(\hat{\mathcal{O}}) \text{ is not a monomial}\};$
- (2) $\left(\bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(f))\right) \cap \mathcal{U}_{\text{loc}}^\lambda;$
- (3) $\left(\bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(\tilde{f}^\lambda))\right) \cap \mathcal{U}_{\text{loc}}^\lambda;$
- (4) $\left(\bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(\widehat{f}^\lambda))\right) \cap \mathcal{U}_{\text{loc}}^\lambda;$
- (5) $\left(\bigcap_{f \in I} \overline{\text{ord}(V(\widehat{f}^\lambda))}\right) \cap \mathcal{U}_{\text{loc}}^\lambda.$

Theorem 4.6. *The following subsets of \mathcal{U}_{loc} coincide:*

- (1) $\{w \in \mathcal{U}_{\text{loc}} \mid \text{For any } f \in I, \text{in}_w(f) \in \text{Gr}^w(\hat{\mathcal{O}}) \text{ is not a monomial}\};$
- (2) $\bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(f));$
- (3) $\bigcup_{\lambda \in \Lambda} \left(\left(\bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(\tilde{f}^\lambda)) \right) \cap \mathcal{U}_{\text{loc}}^\lambda \right);$
- (4) $\bigcup_{\lambda \in \Lambda} \left(\left(\bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(\widehat{f}^\lambda)) \right) \cap \mathcal{U}_{\text{loc}}^\lambda \right);$
- (5) $\bigcup_{\lambda \in \Lambda} \left(\left(\bigcap_{f \in I} \overline{\text{ord}(V(\widehat{f}^\lambda))} \right) \cap \mathcal{U}_{\text{loc}}^\lambda \right).$

Definition 4.7. Let $I \subset \hat{\mathcal{O}}$ be an ideal. We define the local tropical variety of I as

$$\mathcal{T}_{\text{loc}}(I) := \bigcup_{\lambda \in \Lambda} \left(\left(\bigcap_{f \in I} \overline{\text{ord}(V(\widehat{f^\lambda}))} \right) \cap \mathcal{U}_{\text{loc}}^\lambda \right).$$

Proposition 4.8. Let $I \subset \mathbb{C}[x] \subset \hat{\mathcal{O}}$ be an ideal and $I^e = I \cdot \hat{\mathcal{O}}$. Then $\mathcal{T}(I) \cap \mathcal{U}_{\text{loc}}$ and $\mathcal{T}_{\text{loc}}(I^e)$ are equal.

Proof. We show the following two statements:

For each $\lambda \in \Lambda$,

- (i) For $w \in \mathcal{U}_{\text{loc}}^\lambda$, if there exists a $f \in I^e$ s.t. $\text{in}_w(f)$ being a monomial, then there exists a $\tilde{f} \in I$ s.t. $\text{in}_w(\tilde{f})$ being a monomial.
- (ii) For $w \in \mathcal{U}_{\text{loc}}^\lambda$, if there exists a $f \in I$ s.t. $\text{in}_w(f)$ being a monomial, then there exists a $\tilde{f} \in I^e$ s.t. $\text{in}_w(\tilde{f})$ being a monomial.

The proof of the proposition follows from the statements above. The second statement (ii) is clear.

Let $f = \sum_{i=1}^l h_i g_i \in I^e$, where $h_i \in \hat{\mathcal{O}}$ and $g_i \in I$. Suppose $\text{in}_w(f) = m$ is a monomial. Now, for each h_i , we take $\widehat{h_i^\lambda} \in \mathbb{C}[x]$ as same as in Definition 3.8. This implies that $f' := \sum_{i=1}^l \widehat{h_i^\lambda} g_i$ is an element of I .

With the notations above, the following lemma holds: (This Lemma proves statement (i) and completes the proof of proposition.) \square

Lemma 4.9. $\mathcal{T}_h(\text{trop}(f')) \cap \mathcal{U}_{\text{loc}}^\lambda = \mathcal{T}_{\text{loc}}^h(\text{trop}(f)) \cap \mathcal{U}_{\text{loc}}^\lambda$.

Proof. Lemma 3.6 implies that, for each $w \in \mathcal{U}_{\text{loc}}^\lambda$, if h_{ij} is a term of h_i which have the smallest w -weight among the terms of h_i , then it is also an element of $\widehat{h_i^\lambda}$. That means, a term of $\text{in}_w(f)$ is also a term of $f' := \sum_{i=1}^l \widehat{h_i^\lambda} g_i$. We conclude as in the proof of Proposition 3.3. \square

5. LOCAL GRÖBNER FAN

First, we will introduce the local Gröbner fan following Assi, Castro, and Granger [2]. Let $u \in \mathcal{U}_{\text{loc}}$ and define

$$S(u) := \{u' \in \mathcal{U}_{\text{loc}} \mid \text{Gr}^u(\hat{\mathcal{O}}) = \text{Gr}^{u'}(\hat{\mathcal{O}})\}.$$

Then, for a given ideal I in $\hat{\mathcal{O}}$, we define the equivalence relation

$$u \sim u' \Leftrightarrow u' \in S(u) \text{ and } \text{in}_u(I) = \text{in}_{u'}(I).$$

For a local weight vector $u \in \mathcal{U}_{\text{loc}}$, we call $\text{supp}(u) = \{i \mid u_i \neq 0\}$ the support of u . By Lemma 2.1, we have $\text{Gr}^u(\hat{\mathcal{O}}) = \text{Gr}^{u'}(\hat{\mathcal{O}}) \Leftrightarrow \text{supp}(u) = \text{supp}(u')$. So we have:

$$u \sim u' \Leftrightarrow \text{supp}(u) = \text{supp}(u') \text{ and } \text{in}_u(I) = \text{in}_{u'}(I).$$

Definition 5.1. We call the equivalence class:

$$\mathcal{C}[u] := \{u' \in \mathcal{U}_{\text{loc}} \mid u \sim u'\}$$

a local open Gröbner cone (the local Gröbner cone of I w.r.t. u). And we define the set of closures of equivalence classes:

$$\text{LGF}(I) := \{\overline{\mathcal{C}[u]} \mid u \in \mathcal{U}_{\text{loc}}\}$$

as the local Gröbner fan of I .

Theorem 5.2. [3] *Let I be an ideal in $\hat{\mathcal{O}}$. The local Gröbner fan $\text{LGF}(I)$ is a polyhedral fan.*

In this section, we will consider the relation between the local Gröbner fan and the local tropical variety of an ideal in $\hat{\mathcal{O}}$. First, we will state some Propositions.

Proposition 5.3. *Let I be an ideal in $\hat{\mathcal{O}}$. Then the following subsets of \mathcal{U}_{loc} coincide.*

- (1) $\{w \in \mathcal{U}_{\text{loc}} \mid \text{for any } f \in I, \text{in}_w(f) \in \text{Gr}^w(\hat{\mathcal{O}}) \text{ is not a monomial.}\}$
- (2) $\{w \in \mathcal{U}_{\text{loc}} \mid \text{in}_w(I) \subset \text{Gr}^w(\hat{\mathcal{O}}) \text{ contains no monomial.}\}$

The Proposition can be proved with showing that if $\text{in}_w(I)$ contains a monomial, then there exists $f \in I$ such that $\text{in}_w(f)$ is a monomial.

Let I be an ideal in and $\mathcal{C}[w]$ be a local open Gröbner cone of I . For a local vector $w' \in \overline{\mathcal{C}[w]} \setminus \mathcal{C}[w]$, put $\tilde{w} = w' + \epsilon \cdot w$ for some $\epsilon > 0$ sufficiently small.

The following proposition seems to be well-known in the ring of polynomials. For the case of power series, it is called Assi's twin lemma [2]. Since we do not find a proof of this fact in literatures, we will also include a proof.

Proposition 5.4. $\text{in}_{\tilde{w}}(I) = \text{in}_w(\text{in}_{w'}(I))$ in $\text{Gr}^w(\hat{\mathcal{O}})$.

Proof. First, we will show that both $\text{in}_{\tilde{w}}(I)$ and $\text{in}_w(\text{in}_{w'}(I))$ are ideals of the same graded ring $\text{Gr}^{\tilde{w}}(\hat{\mathcal{O}})$ (i.e. $\text{Gr}^w(\text{Gr}^{w'}(\hat{\mathcal{O}})) = \text{Gr}^{\tilde{w}}(\hat{\mathcal{O}})$).

For $\epsilon > 0$, the point $\frac{1}{1+\epsilon}(w' + \epsilon w)$ lies on the open segment (w, w') . Then, we have $w' + \epsilon w \in \mathcal{C}[w]$ by the convexity and properties of cones. (Note that [3] proved the polyhedral property of local Gröbner fan without using this proposition.) This implies $\mathcal{C}[w] = \mathcal{C}[\tilde{w}]$ and consequently $\text{supp}(w) = \text{supp}(\tilde{w})$.

Next, for the proof, we will show the following definition and lemmas:

Definition 5.5. [[3] p7,8, Definition 2.1.3.] Let f be an element of $\hat{\mathcal{O}}$ and \prec be a monomial order. We denote by $\exp_{\prec}(f)$ the maximal element of $\text{Supp}(f)$ w.r.t. \prec and call it leading exponent.

- (1) Let I be an ideal in $\hat{\mathcal{O}}$. We define the set of the leading exponents of I as

$$\text{Exp}_{\prec}(I) = \{\exp_{\prec}(f) \mid f \in I, f \neq 0\}.$$

Then, there exists $\mathcal{G} = \{g_1, \dots, g_r\} \subset I$ such that $\text{Exp}_{\prec}(I) = \bigcup_j (\exp_{\prec}(g_j) + \mathbb{N}^n)$. Such a set \mathcal{G} is called a \prec -standard basis of I .

- (2) Let J be a u -homogeneous ideal (i.e. generated by homogeneous elements) in $\text{Gr}^u(\hat{\mathcal{O}})$. We define the set of the leading exponents of J as

$$\text{Exp}_{\prec}(J) = \{\exp_{\prec}(f) \mid f \in J, f \neq 0\}.$$

Then, there exists $\mathcal{G} = \{g_1, \dots, g_r\} \subset J$ made of homogeneous elements such that $\text{Exp}_{\prec}(J) = \bigcup_j (\exp_{\prec}(g_j) + \mathbb{N}^n)$. Such a set \mathcal{G} is called a (homogeneous) \prec -standard basis of J .

Now given a standard basis G of $J \subset \text{Gr}^u(\hat{\mathcal{O}})$ where $u \in \mathcal{U}_{\text{loc}}$. We say that G is minimal if for $g, g' \in G$, $\exp_{\prec}(g) \in \exp_{\prec}(g') + \mathbb{N}^n$ implies $g = g'$. We say that G is reduced if it is minimal, unitary (i.e. $\text{lc}_{\prec}(g) = 1$ for any $g \in G$) and if for any $g \in G$, $\text{Supp}(g) \setminus \{\exp_{\prec}(g)\} \subset \mathbb{N}^n \setminus \text{Exp}_{\prec}(J)$.

Lemma 5.6. [[3] Lemma 2.1.5.] Given I in $\hat{\mathcal{O}}$ and $u \in \mathcal{U}_{\text{loc}}$, if G is the reduced \prec_u -standard basis then $\text{in}_u(G)$ is the reduced \prec -standard basis of $\text{in}_u(I)$.

Lemma 5.7. [[3] Lemma 2.3.1.] Given an ideal I in $\hat{\mathcal{O}}$ and $u \in \mathcal{U}_{\text{loc}}$, for any $u' \in \overline{C_I[u]} \setminus C_I[u]$ there exists a local order \prec such that the reduced standard bases of I with respect to \prec_u and to $\prec_{u'}$ are the same. (In this proof, we have $\prec = \prec_u^1$, where \prec^1 is a local order satisfying the statements.)

With these lemmas, we will prove Proposition 5.4.

[Proof of Proposition 5.4 (continued)] First, we take $\prec = \prec_w^1$, \prec^1 being a local order. Clearly, $\prec_w = \prec$. Let \mathcal{G} be the \prec_u -reduced standard basis of I . Then, by Lemma 5.6, $\text{in}_w(\mathcal{G})$ is the reduced \prec -standard basis of $\text{in}_w(I)$. And, by Lemma 5.7, \mathcal{G} is also the reduced $\prec_{w'}$ -standard basis of I . By Lemma 5.6, it implies that $\text{in}_{w'}(\mathcal{G})$ is the reduced \prec -standard basis of $\text{in}_{w'}(I)$. Since $\prec_w = \prec$, $\text{in}_w(\text{in}_{w'}(\mathcal{G}))$ is the reduced \prec -standard basis of $\text{in}_w(\text{in}_{w'}(I))$, by Lemma 5.6. Since $w = w' + \epsilon \cdot w$, we have $\text{in}_w(\text{in}_{w'}(\mathcal{G})) = \text{in}_w(\mathcal{G})$. Thus, both $\text{in}_w(I)$ and $\text{in}_w(\text{in}_{w'}(I))$ are ideals of the same ring and have the same basis. This proves the statement. \square

Now we have the following theorem:

Theorem 5.8. *Let I be an ideal in $\hat{\mathcal{O}}$. Then the following subset of \mathcal{U}_{loc} , the local tropical variety of I , is a subfan of the local Gröbner fan of I :*

$$\mathcal{T}_{\text{loc}}(I) = \{w \in \mathcal{U}_{\text{loc}} \mid \text{in}_w(I) \subset \text{Gr}^w(\hat{\mathcal{O}}) \text{ contains no monomials}\}.$$

Proof. Let $\text{LGF}(I) = \{\overline{\mathcal{C}}_1, \dots, \overline{\mathcal{C}}_r\}$ where $\mathcal{C}_1, \dots, \mathcal{C}_r$ are the open Gröbner cones. Suppose that, for any vector w in \mathcal{C}_i ($1 \leq i \leq l$), $\text{in}_w(I)$ contains no monomial, and, for any vector w in \mathcal{C}_j ($l+1 \leq j \leq r$), $\text{in}_w(I)$ contains a monomial. Then for any vector $\overline{\mathcal{C}}_i \setminus \mathcal{C}_i \ni w'$ ($1 \leq i \leq l$), $\text{in}_{w'}(I)$ contains no monomial (proof: $\mathcal{C}_i \ni w = w' + \epsilon a$ for some $a \in \mathbb{R}^n$ and $\epsilon > 0$ is sufficiently small. Then we have $\text{in}_w(I) = \text{in}_a(\text{in}_{w'}(I))$. If $\text{in}_{w'}(I)$ contains a monomial, then $\text{in}_w(I)$ also contains a monomial. This contradicts to hypothesis.). Thus, we have $\mathcal{T}_{\text{loc}}(I) \supset \{\overline{\mathcal{C}}_1, \dots, \overline{\mathcal{C}}_l\}$. Now we suppose that $\text{in}_u(I)$ contains no monomial for some $u \in \overline{\mathcal{C}}_j \setminus \mathcal{C}_j$ ($l+1 \leq j \leq r$). Then, by finiteness of the number of possible initial ideals, we have $u \in \mathcal{C}_i$ for some $1 \leq i \leq l$. Thus we have $\mathcal{T}_{\text{loc}}(I) = \{\overline{\mathcal{C}}_1, \dots, \overline{\mathcal{C}}_l\}$. Since $\text{LGF}(I)$ is a polyhedral fan, the set $\{\overline{\mathcal{C}}_1, \dots, \overline{\mathcal{C}}_l\}$ also satisfies the properties of a polyhedral fan. \square

6. TROPICAL FINITE SET

In this section, we want to find a finite subset \mathcal{H} of an ideal I in $\hat{\mathcal{O}}$ satisfying

$$\bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(f)) = \bigcap_{f \in \mathcal{H}} \mathcal{T}_{\text{loc}}^h(\text{trop}(f)).$$

For this purpose, we will construct a finite subset \mathcal{H} of I satisfying the following condition:

- (*) "For each $w \in \mathcal{U}_{\text{loc}}$, if $w \notin \bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(f))$,
then $\{\text{in}_w(h) \in \text{Gr}^w(\hat{\mathcal{O}}) \mid h \in \mathcal{H}\}$ contains a monomial."

To prove the existence of such a set, let us state lemmas and propositions.

Lemma 6.1. *Let h be in $\hat{\mathcal{O}}$. Then h is w -homogeneous if and only if $\{w \cdot \alpha \mid \alpha \in \text{Supp}(h)\} = \{c\}$ for some $c \in \mathbb{R}$.*

Lemma 6.2. *Given an ideal I in $\hat{\mathcal{O}}$, let $\mathcal{C}[w]$ be a local open Gröbner cone of I . Suppose the dimension of $\mathcal{C}[w] = k \leq n$. Then, for any $h \in \text{Gr}^w(\hat{\mathcal{O}})$, the following statements are equivalent:*

- (1) h is w' -homogeneous, $\forall w' \in \mathcal{C}[w]$;
- (2) h is $(\tilde{w}^1, \dots, \tilde{w}^k)$ -homogeneous, where $\tilde{w}^1, \dots, \tilde{w}^k$ form the 1-skeleton of $\mathcal{C}[w]$;
- (3) h is (w^1, \dots, w^k) -homogeneous, where we suppose $w^1, \dots, w^k \in \mathcal{C}[w]$ are independent over \mathbb{R} .

Proof. It is clear that (1) implies (2).

(3) \Rightarrow (1): By the hypothesis and Lemma 6.1, there exists $c_i \in \mathbb{R}$ s.t. $\{w^i \cdot \alpha \mid \alpha \in \text{Supp}(h)\} = \{c_i\}$ for each $1 \leq i \leq k$. Fix $w' \in \mathcal{C}[w]$. There exist $a_i \in \mathbb{R}$ s.t. $w' = \sum_{i=1}^k a_i w^i$. For any $\alpha \in \text{Supp}(h)$, we have $\alpha \cdot w' = \sum_{i=1}^k a_i (w^i \cdot \alpha) = \sum_{i=1}^k a_i c_i \in \mathbb{R}$. By Lemma 6.1 again, we have h is w' -homogeneous.

(2) \Rightarrow (1): By the hypothesis, for $w' \in \mathcal{C}[w]$ there exist $a_i \in \mathbb{R}_{>0}$ s.t. $w' = \sum_{i=1}^k a_i \tilde{w}^i$. Then the proof is similar to above proof.

(1) \Rightarrow (2): Now we will prove that (1) implies the following statement (which is more general than (2)): for any $\tilde{w} \in \overline{\mathcal{C}[w]} \setminus \mathcal{C}[w]$, h is \tilde{w} -homogeneous. Let $\{w(l) \in \mathcal{C}[w] \mid l \in \mathbb{N}\}$ be a sequence of $\mathcal{C}[w]$ s.t. $\lim_{l \rightarrow \infty} w(l) = \tilde{w}$. Lemma 6.1 implies that, for any $l \in \mathbb{N}$ we have $w(l) \cdot (\alpha - \alpha') = 0$ where $\alpha, \alpha' \in \text{Supp}(h)$ are arbitrary. The continuity of $x \mapsto x \cdot (\alpha - \alpha')$ implies $\tilde{w} \cdot (\alpha - \alpha') = 0$. Then $\{\tilde{w} \cdot \alpha \mid \alpha \in \text{Supp}(h)\} = \{c\}$ for some $c \in \mathbb{R}$ and, again, by Lemma 6.1, h is \tilde{w} -homogeneous. \square

Proposition 6.3. *Let I be an ideal in $\hat{\mathcal{O}}$ and $\mathcal{C}[w]$ be a local open Gröbner cone of I . Suppose that, for $w \in \mathcal{C}[w]$, $\text{in}_w(I) \subset \text{Gr}^w(\hat{\mathcal{O}})$ contains a monomial m . Then there exist $f \in I$ s.t. $\text{in}_{w'}(f) = m$ for any $w' \in \mathcal{C}[w]$.*

Let the local Gröbner fan of I be

$$\text{LGF}(I) = \{\overline{\mathcal{C}_1[w_1]}, \dots, \overline{\mathcal{C}_r[w_r]}\} \subset \mathcal{U}_{\text{loc}},$$

where $\mathcal{C}_1[w_1], \dots, \mathcal{C}_r[w_r]$ are the open Gröbner cones. Suppose that, for any $1 \leq i \leq l$, $\text{in}_{w_i}(I) \subset \text{Gr}^{w_i}(\hat{\mathcal{O}})$ contains no monomial, and, for each $l+1 \leq j \leq r$, $\text{in}_{w_j}(I) \subset \text{Gr}^{w_j}(\hat{\mathcal{O}})$ contains a monomial m_j . Then, by Proposition 6.3, for each m_j we can find $f_j \in I$ s.t. $\text{in}_{w'_j}(f_j) = m_j$ for any $w'_j \in \mathcal{C}[w_j]$. We define $\mathcal{H} = \{f_j \mid l+1 \leq j \leq r\}$.

Proposition 6.4. $\mathcal{H} = \{f_j \mid l+1 \leq j \leq r\}$ is a local tropical finite set of I , i.e. satisfies $(*)$.

Proof. By Theorem 4.6, Proposition 5.3 and Proposition 5.8, the following subsets of \mathcal{U}_{loc} coincide:

- (1) $(\bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(f)))$;
- (2) $\{w \in \mathcal{U}_{\text{loc}} \mid \text{for any } f \in I, \text{ in}_w(f) \in \text{Gr}^w(\hat{\mathcal{O}}) \text{ is not a monomial}\}$;
- (3) $\{w \in \mathcal{U}_{\text{loc}} \mid \text{in}_w(I) \subset \text{Gr}^w(\hat{\mathcal{O}}) \text{ contains no monomial}\}$;
- (4) $\{\overline{\mathcal{C}_1[w_1]}, \dots, \overline{\mathcal{C}_l[w_l]}\}$.

Let $w \notin \bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(f))$, then $w \in \{\mathcal{C}_{l+1}[w_{l+1}], \dots, \mathcal{C}_r[w_r]\}$ and the set $\{\text{in}_w(h) \mid h \in \mathcal{H}\}$ contains a monomial. \square

Now we have following theorem:

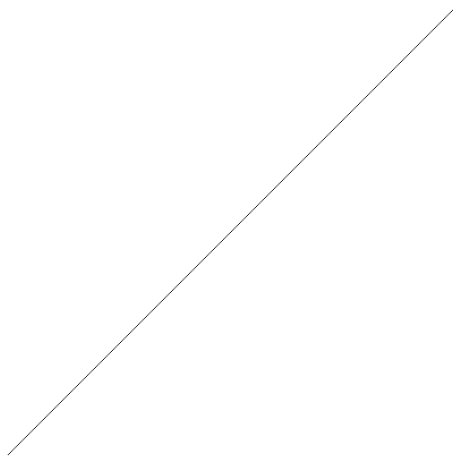
Theorem 6.5. Given an ideal I in $\hat{\mathcal{O}}$, the following subsets of \mathcal{U}_{loc} coincide:

- (1) $\{w \in \mathcal{U}_{\text{loc}} \mid \text{for any } f \in I, \text{ in}_w(f) \in \text{Gr}^w(\hat{\mathcal{O}}) \text{ is not a monomial}\}$;
- (2) $\{w \in \mathcal{U}_{\text{loc}} \mid \text{in}_w(I) \subset \text{Gr}^w(\hat{\mathcal{O}}) \text{ contains no monomial}\}$;
- (3) $\bigcap_{f \in I} \mathcal{T}_{\text{loc}}^h(\text{trop}(f))$;
- (4) $\mathcal{T}_{\text{loc}}(I)$;
- (5) $\{\overline{\mathcal{C}_1[w_1]}, \dots, \overline{\mathcal{C}_l[w_l]}\}$;
- (6) $\bigcap_{l+1 \leq j \leq r} \mathcal{T}_{\text{loc}}^h(\text{trop}(f_j))$.

Finally, we will finish this paper with an example.

Example 6.6. Let $f_1 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$ be the Maclaurin's expansion of e^x . And let $f = y(f_1 - 1) - x^2 = y(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots - 1) - x^2 = xy - x^2 + \frac{x^2y}{2} + \frac{x^3y}{6} + \dots$. Now we will compute the local tropical variety of $\langle f \rangle$.

First, as in Definitions 3.2 and 3.8, for $\mathcal{U}_{\text{loc}}^0 = \{(w_1, w_2) \mid w_1, w_2 > 0\}$ we have $\widehat{f^0} = \widehat{f^0} = xy - x^2$, for $\mathcal{U}_{\text{loc}}^1 = \{(0, w_2) \mid w_2 > 0\}$ we have $\widehat{f^1} = -x^2$, for $\mathcal{U}_{\text{loc}}^2 = \{(w_1, 0) \mid w_1 > 0\}$ we have $\widehat{f^2} = xy$, and for $\mathcal{U}_{\text{loc}}^{12} = (0, 0)$ we have $\widehat{f^{12}} = 0$. By Definition 3.14, the local tropical variety of $\langle f \rangle$ is as in figure 1.

FIGURE 1. the local tropical variety of f

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DEPARTMENT OF MATHEMATICS, KOBE UNIVERSITY, ROKKO 1-1, KOBE 657-8501,
JAPAN

E-mail address: `touda@math.kobe-u.ac.jp`